Assignment 2.

This homework is due *Thursday*, September 13.

There are total 26 points in this assignment. 23 points is considered 100%. If you go over 23 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 1.3 and a part of 2.1 in Bartle–Sherbert. (Until I get the new edition, exercise numbers refer to 3rd edition.)

1. Quick cheat-sheet

Reminder. (Subsection 2.1.1) On the set $\mathbb{R}$ of real numbers there two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $a+b=b+a$ for all $a,b\in\mathbb{R}$,
(A2) $(a+b)+c=a+(b+c)$ for all $a,b,c\in\mathbb{R}$,
(A3) there exists $0\in\mathbb{R}$ s.t. $0+a=a+0=a$ for all $a\in\mathbb{R}$,
(A4) for each $a\in\mathbb{R}$ there exists an element $-a$ s.t. $a+(-a)=(-a)+a=0$,
(M1) $ab=ba$ for all $a,b\in\mathbb{R}$,
(M2) $(ab)c=a(bc)$ for all $a,b,c\in\mathbb{R}$,
(M3) there exists $1\in\mathbb{R}$ s.t. $1\cdot a=a\cdot 1=a$ for all $a\in\mathbb{R}$,
(M4) for each $a\neq 0$ in $\mathbb{R}$ there exists an element $1/a$ s.t. $a\cdot(1/a)=(1/a)\cdot a=1$,
(D) $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for all $a,b,c\in\mathbb{R}$.

2. Exercises

(1) (Parts of 2.1.1, 2, 5) For $a,b\in\mathbb{R}$, prove that
  (a) [1pt] $-(a+b)=-a+(-b)$, (*Hint:* show that the number $-a+(-b)$ satisfies the definition of a number opposite to $(a+b)$ in (A4).)
  (b) [1pt] $-(-a)=a$,
  (c) [1pt] $-(a/b)=(-a)/b$ if $b\neq 0$,
  (d) [1pt] if $a\neq 0,\ b\neq 0$, then $1/(ab)=(1/a)(1/b)$.

  Every equality you write should be accompanied by a reference to the exact property of real numbers or theorem you are using.

(2) [1pt] Prove that if $a\in\mathbb{R}$ satisfies $a\cdot a=a$, then $a=0$ or $a=1$.

(3) (a) [1pt] Is addition of real numbers distributive over multiplication?
  (b) [1pt] Is set union distributive over set intersection? That is, is it true that $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ for all sets A,B,C?
  (c) [1pt] Is set intersection distributive over set union?

— *see next page* —

(4) On the set $\mathbb{N}$, consider two operations: $\oplus$ and $\odot$ defined as follows: $a \oplus b = ab$ and $a \odot b = a^b$.

(a) [1pt] Do properties A1, A2 hold for $\oplus$? That is, is it true that $a \oplus b = b \oplus a$, and that $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in \mathbb{N}$?
(*Hint:* For this and further items, the main way to figure out questions is to write out expressions with $\odot$ and $\oplus$ in terms of "usual" operations, using definition of $\odot$ and $\oplus$.)

(b) [1pt] Do properties M1, M2 hold for $\odot$?

(c) [1pt] Is there unit element with respect to $\odot$? That is, is there an element $1_\odot \in \mathbb{N}$ such that $1_\odot \odot a = a \odot 1_\odot = a$ for all $a \in \mathbb{N}$?

(d) [1pt] Is there a *right* unit element with respect to $\odot$? That is, is there an element $1_r \in \mathbb{N}$ such that $a \odot 1_r = a$ for all $a \in \mathbb{N}$?

(e) [1pt] Is there a *left* unit element with respect to $\odot$? That is, is there an element $1_\ell \in \mathbb{N}$ such that $1_\ell \odot a = a$ for all $a \in \mathbb{N}$?

(f) [1pt] Is $\odot$ distributive over $\oplus$ *on the left*? That is, is it true that $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$?

(g) [1pt] Is $\odot$ distributive over $\oplus$ *on the right*? That is, is it true that $c \odot (a \oplus b) = (c \odot a) \oplus (c \odot a)$?

(5) [2pt] (1.3.5) Exhibit (define explicitly) a bijection from N to the set of all odd integers greater than 13.

(6) Exhibit (define explicitly) a bijection between

(a) [2pt] $\mathbb{Z}$ and $\mathbb{Z} \setminus \{0\}$,

(b) [3pt] $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$,

(7) [4pt] (1.3.12) Prove that the collection $\mathcal{F}(\mathbb{N})$ of all *finite* subsets of $\mathbb{N}$ is countable.